Discussion 4 Worksheet

Vectors

Date: 9/3/2021

MATH 53 Multivariable Calculus

1 Dot Products

1. If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$ and $u \circ v = -1$, what is the angle between $\vec{u}$ and $\vec{v}$?
2. Find three nonzero vectors in $\mathbb{R}^3$ that are perpendicular to $\langle 1, 3, 2\rangle$.
3. Let P be a vertex on a cube. Let Q be an adjacent vertex and let R be the vertex opposite to P. Using dot products, find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
4. If $\vec{u}$ and $\vec{v}$ are unit vectors in $\mathbb{R}^3$, show that the vectors $\vec{u} + \vec{v}$ and $\vec{v} - \vec{v}$ are perpendicular.
5. Derive the *polarization identity*: if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, then $\vec{u} \circ \vec{v} = \frac{1}{4}\left(|\vec{u} + \vec{v}|^2 - |\vec{u} - \vec{v}|^2\right)$. Hint: it is simplest not to work straight from the definition of the dot product (although this will work too).

2 Challenge: Parallelogram Law

Consider a parallelogram with side lengths a and b, and diagonals of lengths c and d. Show that $2a^2 + 2b^2 = c^2 + d^2$. Hint: use vector geometry and dot products.

3 Vector and Scalar Projections

1. For each of the following pairs of vectors, find the vector projection of $\vec{v}$ onto $\vec{w}$ and the scalar projection of $\vec{v}$ onto $\vec{w}$.
   a) $\vec{v} = \langle 2, 4\rangle$, $\vec{w} = \langle 3, 1\rangle$.
   b) $\vec{v} = \langle 5, -1\rangle$, $\vec{w} = \langle 2, 9\rangle$.
   c) $\vec{v} = \langle -6, 3, 2\rangle$, $\vec{w} = \langle 1, -5, 3\rangle$.
2. Find formulas for the vector and scalar projections of a vector $\vec{v}$ onto $\vec{w}$ involving the cosine of the angle θ between $\vec{v}$ and $\vec{w}$.

4 Cross Product Computations

Find the cross products $\vec{v} \times \vec{w}$ of the following pairs of vectors.

1. $\vec{v} = \langle 2, 3, 1\rangle$, $\vec{w} = \langle -1, 2, 3\rangle$.
2. $\vec{v} = 6\vec{i} - 4\vec{j} - 3\vec{j}$, $\vec{w} = 4\vec{i} + \vec{j}$.
3. $\vec{v}$ pointing a distance 5 units in the positive x-direction, $\vec{w}$ a unit vector lying in the first quadrant of the xy-plane and making an angle of $\pi/4$ with the x-axis.

5 Cross Product Concepts and Applications

1. Given vectors $\vec{v}$ and $\vec{w}$, find an identity which relates the four quantities $|\vec{v}|$, $|\vec{w}|$, $|\vec{v} \times \vec{w}|$, and $|\vec{v} \cdot \vec{w}|$. (Hint: Consider any relevant trigonometric identities.)

2. Let $\vec{u}$ and $\vec{v}$ be nonzero vectors with $\vec{u} \times \vec{v} = \vec{0}$. What can you say about the relationship between $\vec{u}$ and $\vec{v}$?

3. Find the area of the triangle with two sides given by the vectors $\vec{v} = \langle 1, 2 \rangle$ and $\vec{w} = \langle -3, 4 \rangle$.

6 Challenge: BAC-CAB

Prove the "BAC-CAB" / "double-crossing" rule:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

(NOTE: Typically scalars such as $\vec{a} \cdot \vec{c}$ are written on the left in scalar multiplication. This formula is a rare exception and is written this way because "BAC-CAB" is easier to remember than "ACB-ABC.")

7 True/False

Supply convincing reasoning for your answer.

(a) T F If you take a cross product of two vectors lying in the xy-plane, your result will point along the z-axis.

(b) T F The cross product makes sense for vectors in any number of dimensions.

(c) T F The absolute value of the scalar projection of a vector $\vec{v}$ onto another vector $\vec{w}$ is equal to the norm of the vector projection of $\vec{v}$ onto $\vec{w}$.

(d) T F The cross product is associative: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

(e) T F The dot and cross products satisfy $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b}) \times \vec{c}$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.